On the Form Factors of Relevant Operators and their Cluster Property

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Abstract

We compute the Form Factors of the relevant scaling operators in a class of integrable models without internal symmetries by exploiting their cluster properties. Their identification is established by computing the corresponding anomalous dimensions by means of Delfino–Simonetti–Cardy sum–rule and further confirmed by comparing some universal ratios of the nearby non–integrable quantum field theories with their independent numerical determination.

1 Introduction

In this work we present a detailed investigation of the matrix elements

$$F^{\mathcal{O}}_{a_1,a_2,\ldots,a_n}(\beta_1,\ldots,\beta_n) \;=\; \langle 0 \mid \mathcal{O}(0) \mid A_{a_1}(\beta_1)\ldots A_{a_n}(\beta_n)\rangle \qquad (1.1)$$

(the so–called Form Factors (FF)) in a class of integrable two–dimensional quantum field theories. Our specific aim is to check some new theoretical ideas which concern the relationships between three different regimes which two–dimensional quantum field theories may have, namely the ones ruled by conformal invariance, integrable or non–integrable dynamics.

Conformal Field Theories (CFT) and the associated off-critical Integrable Models (IM) have been extensively studied in the last years: as a result of these analyses a good deal of information has been obtained particularly on correlation functions of a large number of statistical mechanical models in their scaling limit and on physical quantities related to them (see for instance [1-8]). In this context, a crucial problem often consists in the determination of the spectrum of the scaling operators away from criticality, namely their correct identification by means of the set of their Form Factors. This is one of the issues addressed in this work.

Form Factors also play a crucial role in estimating non–integrable effects. Let us first recall that the above CFT and IM regimes cannot obviously exhaust all possible behaviours that statistical models and quantum field theories can have since typically they do not possess an infinite number of conservation laws. This means that in general we have to face all kinds of phenomena and complications associated to Non–Integrable Models (NIM). The scattering amplitudes and the matrix elements of the quantum operators will have in these cases a pattern of analytic singularities due both to the presence of higher thresholds and to the appearance of resonances. A first step forward in their analysis has been recently taken in ref. [9] where it has been shown that some interesting examples of NIM may be obtained as deformations of integrable models. The action of such theories can correspondingly be written as

$$\mathcal{A} \;=\; \mathcal{A}_{int} + \sum_i \lambda_i \int d^2x \, \Psi_i(x) \;, \qquad (1.2)$$

$\mathcal{A}_{int}$ being the action of the integrable model. Since the exact expressions (1.1) of the Form Factors of the integrable theories are all assumed calculable, in particular the ones of the fields $\Psi_i(x)$ entering eq. (1.2), one is inclined to study the non–integrable effects by using the Born series based on the Form Factors. Although at first sight this still remain a difficult task (and generally, it is indeed so), there may be favorable circumstances where the analysis simplifies considerably. For instance, as far as there is only a soft breaking of integrability, it has been shown in [9] that the complications of the higher terms in the series can often be avoided since the most relevant corrections only come from the lowest approximation. If this is the case, one can extract an important amount of information with relatively little effort: a significant set of physical quantities to look at is provided for instance by universal ratios, like the ones relative to the variations of the masses or of the vacuum energy density $\mathcal{E}_{vac}$: if the breaking of integrability is realized by means of a single field $\Psi(x)$, those are expressed by

$$
\frac{\delta m_i}{\delta m_j} = \frac{m_j^{(0)}}{m_i^{(0)}} \frac{F_{ii}^{\Psi}(i\pi)}{F_{jj}^{\Psi}(i\pi)} ,
$$

$$
\frac{\delta \mathcal{E}_{vac}}{m_1^{(0)} \delta m_1} = \frac{\langle 0 \mid \Psi \mid 0 \rangle}{F_{11}^{\Psi}(i\pi)} ,
$$

(1.3)

where $m_i^{(0)}$ refers to the (unperturbed) mass spectrum of the original integrable theory. It is thus evident that also to estimate the non–integrable effects associated to a given operator $\Psi(x)$ one must face the problem of correctly identifying its FF's.

Two new results on the relationship between CFT and IM have been recently derived by Delfino, Simonetti and Cardy [10]. Briefly stated, the first result consists in a new sum–rule which relates the conformal dimension Δ^ϕ of the operator $\phi(x)$ to the off–critical (connected) correlator $\langle \Theta(x)\phi(0) \rangle_c$, where $\Theta(x)$ is the trace of the stress–energy tensor[1]

$$
\Delta^\phi = -\frac{1}{4\pi \langle \phi \rangle} \int d^2x \, \langle \Theta(x)\phi(0) \rangle_c .
$$

(1.4)

This sum–rule is closely related to the analogous expression for the conformal central

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[1]The sum–rule in the form of eq. (1.4) may be violated by effect of renormalization of the operators outside the critical point, as clarified in the original reference [10]. This is however not the case for the field theories and the operators considered in this work.

charge c [11]

$$c = \frac{3}{4\pi} \int d^2x \mid x \mid^2 \langle\Theta(x)\Theta(0)\rangle_c \ . \tag{1.5}$$

Equations (1.4) and (1.5) express elegant relationships between conformal and off–critical data, but more importantly, they provide very concrete and efficient tools to characterise the scaling limit of the off-critical models.

As for the second result, it has been suggested by the aforementioned authors of ref. [10], that the Form Factors of the relevant scaling fields[2] of an integrable field theory – in absence of internal symmetries – are in one–to–one correspondence with the independent solutions of the so called *cluster equations*

$$\lim_{\Lambda\to\infty} F^{\Phi}_{a_1,a_2,\ldots,a_k,a_{k+1},\ldots,a_{k+l}}(\theta_1,\theta_2,\ldots,\theta_k,\Lambda+\theta_{k+1},\ldots,\Lambda+\theta_{k+l}) \quad = \tag{1.6}$$

$$= \frac{1}{\langle\Phi\rangle} F^{\Phi}_{a_1,a_2,\ldots,a_k}(\theta_1,\theta_2,\ldots,\theta_k)\, F^{\Phi}_{a_{k+1},\ldots,a_{k+l}}(\theta_{k+1},\ldots,\theta_{k+l})$$

These equations can be imposed on the Form Factors *in addition* to the customary functional and residue equations which they satisfy (see in this respect also [1, 12]). If this *cluster hypothesis* is valid, we would have a clear method to identify the matrix elements of all the relevant operators, at least in the case of theories without symmetries. It must be stressed that until now this task has often been a matter of keen guess–work and mostly based on physical intuition.

It turns out that a check of the above *cluster hypothesis* provides a well–suited forum for testing several theoretical aspects. In fact, the most direct way of confirming the above idea is firstly to solve the general functional equations of the Form Factors with the additional constraints of the cluster equations (1.6) and to see whether *the number of independent solutions equals the number of relevant fields* in the corresponding Kac table. If the above check turns out to be positive, one may use the sum–rule (1.4) in order to achieve the correct identification of the (supposed) primary relevant operators ϕ_i: from the values of the partial sums one can in fact infer the value of the anomalous dimension and correspondingly recognize the operator. An additional confirmation may also come from the employment of eqs. (1.3) relative to non–integrable field theories. In

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[2]Hereafter we are using the short expression "scaling fields" to actually denote the off-critical operators which reduce to the scaling fields in the conformal limit.

fact, one can regard the primary field $\phi_i(x)$ under investigation as that operator which spoils the integrability of the original theory and therefore compare the predictions (1.3) based on its Form Factors with their independent numerical determinations which may be obtained by means of the truncation method [13]. Note that a successful test of this kind could also be interpreted the other way around, namely as a further proof of the effectiveness of the formulas (1.3) in estimating non–integrable effects.

The models on which we have chosen to test the above considerations are integrable deformations of the first representatives of the non–unitary conformal series[3] $\mathcal{M}(2, 2\, n + 1)$, $n \geq 2$. They belong to the class of universality of solvable RSOS lattice models *à la* Andrews–Baxter–Forrester although with negative Boltzmann weigths [14, 15]: their simplest example is given by the quantum field theory associated to the so–called Yang–Lee model which describes the distribution of zeros in the grand canonical partition function of the Ising model in a complex magnetic field [16, 17]. These models do not have any internal symmetry and all their fields are relevant operators: hence, they are ideal for our purposes. Moreover, the nature of their massive and conformal phases is simple enough. The price to pay for their relative simplicity is however the presence of typical non–unitary phenomena, as imaginary coupling constants or negative values of the anomalous dimensions and central charge, together with the anomalous poles in the S–matrix which induce an unusual analytic structure in the Form Factors [4, 17, 18].

The paper is organized as follows. In Section 2 we discuss the general strategy which can be employed in order to compute the FF's of the relevant operators in the integrable deformations of the models $\mathcal{M}(2, 2\, n+1)$. In Section 3 and 4 we present a detailed analysis of the FF's of the models $\mathcal{M}(2, 7)$ and $\mathcal{M}(2, 9)$, which are the first non–trivial examples on which to check all the theoretical ideas discussed above. In fact, for the first model

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[3]The conformal weigths and central charge are given respectively by

$$\Delta_{1,a} = -\frac{(a - 1)\,(2n - a)}{2\,(2n + 1)}, \quad a = 1, 2, \ldots, 2n$$

$$c = -\frac{2\,(6n^2 - 7n + 1)}{2n + 1}, \quad n = 2, 3, \ldots$$

$\mathcal{M}(2,5)$ the *cluster hypothesis* is easily verified: the only solution of the Form Factor equations is the sequence of functions determined in ref. [7] which indeed fulfill the cluster equations (1.6) and are easily identified with the matrix elements of the only relevant field of the Yang–Lee model. The two models $\mathcal{M}(2,7)$ and $\mathcal{M}(2,9)$ represent, somehow, the best playground for our purposes because they give rise to integrable models under each of their possible (individual) deformations and also because they optimize the size of the lenghty numerical output which we present for the solutions of the non–linear equations. Moreover, although there is in principle no obstacle to extend the analysis to all the models $\mathcal{M}(2,2\,n+1)$, these are the simplest cases from a computational point of view since the larger is the value of the index n the higher is the order of the system of algebraic equations to be solved for determining the Form Factors. Finally, our conclusions are in Section 5. Two appendices complete the paper: Appendix A gathers all important formulas relative to the parameterization of the two–particle Form Factors and Appendix B collects the S–matrices of the models analysed.

2   Outline of Our Strategy

In this section we discuss the general strategy needed in order to obtain the Form Factors of the scaling primary fields of the integrable deformations $\phi_{1,k}$ of the conformal models $\mathcal{M}(2,2n+1)$ (hereafter denoted by the shorthand notation $[\mathcal{M}(2,2n+1)]_{(1,k)}$). The deforming field $\phi_{1,k}$ can be one of the operators $\phi_{1,2}$, $\phi_{1,3}$ or possibly some other primary field which gives rise to an integrable deformation.

The starting point in the computation of the Form Factors is the correct parameterization of the two–particle ones which is given detailed in Appendix A. This is a non–trivial task in the case of non–unitary models for the reason that the exact S–matrices of these models are usually plagued by a pletora of anomalous poles [18]. By this we mean for example simple poles which are not related to any bound state, or, more generally, any poles which do not have apparently the standard diagrammatic interpretation of refs. [19]. Consider for example the S–matrices listed in the tables of Appendix B relative to the integrable deformations of the models $\mathcal{M}(2,7)$ and $\mathcal{M}(2,9)$ where the anomalous poles

have been labelled with $\mathcal{B}$, $\mathcal{D}$ or $*$. The origin of these poles may be explained according to the ideas put forward in [20]. In particular, poles of type $\mathcal{B}$ and $\mathcal{D}$ are due to multi-particle processes of the kind described respectively by the "butterfly" and "dragonfly" diagrams drawn in Figures 2 and 3 respectively. These multi–loop processes induce in the S–matrix simple poles rather than higher order ones because the internal lines of these diagrams cross at relative rapidity values relative to some zeros of their corresponding two–particle S–matrix element: this gives rise to a partial cancellation of the poles.

The adopted parameterization for two–particle FF's is directly related to the pole structure of the S–matrix. This yields to the expression (A.4) whose functional form is set except for the coefficients $a_{ab,\Phi}^{(k)}$ appearing in the expansion (A.8) of the polynomials $Q_{ab}^{\Phi}(\theta)$. The degree $k_{ab,\Phi}^{\max}$ of these polynomials is fixed by the asymptotic behavior of the FF's for large rapidities which depends, of course, on the field Φ [3]. For the case of two–particle FF's of cluster operators, it is easy to see that they are subject to have for large θ at most a constant limit[4]. In fact, for two–particle FF's eqs. (1.6) read

$$\lim_{\theta \to \infty} F_{ab}^{\Phi}(\theta) = F_a^{\Phi} F_b^{\Phi} \ . \tag{2.1}$$

Hereafter we deal with dimensionless cluster operators which are normalized in such a way as to have a vacuum expectation value equal to one[5]

$$\langle 0|\Phi(0)|0\rangle = F_0^{\Phi} = 1 \ . \tag{2.2}$$

In order to fully determine the FF's of the cluster operators we have chosen to focus on the set of all one– and two–particle FF's. Listing all the relations among them, one obtains a system of equations in the unknown parameters F_a^{Φ} and $a_{ab,\Phi}^{(k)}$. Let us see then all information we have on the FF's.

The first equations that one must consider are the *dynamical residue equations* resulting from the detailed analysis of the poles they are endowed with. These equations relate FF's with different external particles and may have a different origin. In particular,

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[4]The limit may vanish in the presence of symmetries.

[5]Since the relevant primary operators will be identified with the cluster ones (except their dimensional factors which can be easily restored), in the sequel we will adopt the same normalization also for them.

for every simple bound state pole of the amplitude S_{ab} at angle $\theta = iu_{ab}^c$ relative to the particle A_c (see Figure 1), we have

$$\lim_{\theta \to iu_{ab}^c} (\theta - iu_{ab}^c) F_{ab}^\Phi(\theta) = i \, \Gamma_{ab}^c F_c^\Phi \ , \tag{2.3}$$

where the on–mass–shell three–point coupling constant Γ_{ab}^c is given by the residue on the pole of the S–matrix

$$\lim_{\theta \to iu_{ab}^c} (\theta - iu_{ab}^c) S_{ab}(\theta) = i \, (\Gamma_{ab}^c)^2 \ . \tag{2.4}$$

Dynamical residue equations are also provided by double order poles and simple order poles of type $\mathcal{B}$. Both of them are related to diagrams of the kind shown in Figure 2. For each such diagram, one can write the following equation

$$\lim_{\theta_{ab} \to i\varphi} (\theta_{ab} - i\varphi) F_{ab}^\Phi(\theta_{ab}) \; = \; i \, \Gamma_{ad}^c \, \Gamma_{db}^e \, F_{ce}^\Phi(i\gamma) \ , \tag{2.5}$$

where $\gamma = \pi - u_{cd}^a - u_{de}^b$. In the case of $\mathcal{B}$ poles one can always verify that the amplitude $S_{ce}(\theta)$ has a simple zero at $\theta = i\gamma$. More complicated residue equations can be in general obtained with reference to $\mathcal{D}$ poles and higher order ones whose explicit expressions – not reported here – can be however easily written, once the corresponding multi–scattering diagrams have been identified.

It must be stressed that the above set of equations just depend on the dynamics of the model through its S–matrix and hold identical for every operator $\Phi(x)$. Therefore, in general, some residual freedom on the parameters is still expected after imposing these equations, because they must be satisfied by the FF's of all operators compatible with the assumed asymptotic behaviour.

Adding to this system of *linear* equations the *non–linear* cluster equations (2.1) of the two–particle FF's, one obtains in general a redundant set of compatible equations in all the unknown parameters of the one– and two–particle FF's. Due to its non–linearity, the system allows a multiplicity of solutions which define the so–called *cluster operators* of the theory[6]. If the number of solutions of the system matches the cardinality of the Kac

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[6]In all cases analyzed, the smallest system of equations among different FF's which is sufficient to determine their coefficients turns out to involve just a subset of the two–particle FF's. This suggests that also in the general case it should be possible to predict the final number of cluster solutions already from

table of the model one is led to identify them with the families of FF's of the relevant primaries.

Among the cluster solutions, one can first of all identify the FF's of the deforming field $\phi_{1,k}$. This operator is known to be essentially the trace of the energy–momentum tensor $\Theta(x)$ since

$$\Theta(x) = 4\pi \, \mathcal{E}_{vac} \, \phi_{1,k} \,, \tag{2.6}$$

$\mathcal{E}_{vac}$ being the vacuum energy density which can be easily computed by TBA computations [21]

$$\mathcal{E}_{vac} = -\frac{m_1^2}{8 \sum_{x \in P_{11}} \sin(\pi x)} \,. \tag{2.7}$$

Here the set P_{11} is defined in eq. (A.1) and m_1 is the lightest particle mass. In view of the proportionality (2.6), the FF's of $\phi_{1,k}$ can be identified among the cluster solutions by checking the peculiar equations which characterize the two–particle FF's of $\Theta(x)$ in virtue of the conservation of the energy–momentum tensor, namely the normalization of the diagonal two–particle FF's

$$F_{aa}^\Theta(i\pi) = 2\pi m_a^2 \,, \tag{2.8}$$

and the factorization of the polynomial Q_{ab}^Θ for non–diagonal two–particle FF's $(a \neq b)$ into

$$Q_{ab}^\Theta(\cosh\theta) = \left(2\, m_a \, m_b \, \cosh\theta + m_a^2 + m_b^2 \right) R_{ab}^\Theta(\cosh\theta) \,, \tag{2.9}$$

where R_{ab}^Θ is a suitable polynomial [1, 3].

Knowing the FF's of $\Theta(x)$, one is then enabled to make use of the sum–rule (1.4) to compute the conformal dimension of the operators defined by the remaining cluster solutions in order to identify them with all the relevant primaries of the theory. This sum–rule can be evaluated by using the spectral representation of the correlator

$$\langle \Theta(x)\phi(0) \rangle_c = \sum_{n=1}^\infty \sum_{a_i} \int_{\theta_1 > \theta_2 \ldots > \theta_n} \frac{\mathrm{d}^n\theta}{(2\pi)^n} \, F_{a_1,\ldots,a_n}^\Theta(\theta) \, F_{a_1,\ldots,a_n}^\phi(i\pi - \theta) \, e^{-|x| \sum_{k=1}^n m_k \cosh\theta_k} \,. \tag{2.10}$$

In all the models we have studied, the corresponding series for the sum–rule (1.4) displays a very fast convergence behaviour for any of the cluster operators. The truncated sums

a "minimal" system, avoiding in this way to deal with systems of equations involving a huge number of unknown variables.

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obtained by including just very few contributions have proved sufficient to attain a good approximation of all the values expected by the Kac table of conformal dimensions. In this way, the one–to–one correspondence between cluster solutions and primary relevant operators can been easily set.

Finally, having obtained the FF's of all the relevant fields in each integrable deformation, as a further check of their correct identification, one may employ the formulas (1.3) relative to the universal ratios of the nearby non–integrable quantum field theories. These predictions can then be compared against their numerical estimates obtained from Truncated Conformal Space (TCS) approach developed in [13]. The agreement between numerical estimates and theoretical predictions of the non–integrable effects may provide additional confirmation and may remove all possible remaining doubts about the validity of the cluster hypothesis for these models.

3   Integrable Deformations of $\mathcal{M}(2,7)$

The minimal conformal model $\mathcal{M}(2,7)$ has, in addition to the identity operator $\phi_{1,1}$, only two primary operators, $\phi_{1,2}$ and $\phi_{1,3}$, both of them relevant with conformal weights given by $-2/7$ and $-3/7$ respectively [4]. The perturbations of the conformal action either by the "magnetic operator" $\phi_{1,2}$ or by the "thermal operator" $\phi_{1,3}$ are both known to be, separately, integrable [18]. The S–matrices and the mass ratios of the two integrable models are given in tables B1 and B2. In their massive phase, both perturbations have two stable massive particles denoted by A_1 and A_2, with a mass ratio and a scattering matrix which depend on the integrable direction considered. In each case, we expect to find two non–trivial independent families of Form Factors solutions to the cluster equations (1.6) (in addition to the family of the null Form Factors relative to the identity operator).

The Form Factors of the primary operators of the model relative to the thermal deformation have already been considered in [8]. Here, we have performed an *ab–initio* calculation by imposing the cluster equations: our result has been in perfect agreement with the FF's of ref. [8], proving in this way that these cluster solutions are also unique.

The result of the computation of Form Factors in the two integrable deformations $[\mathcal{M}(2,7)]_{(1,2)}$ and $[\mathcal{M}(2,7)]_{(1,3)}$ are summarised in tables 1–2 and 3–4 respectively where we list the values of the one–particle FF's and the coefficients $a_{ab,\phi}^{(k)}$ of the two–particle FF's relative to some of the lightest two–particle states. As expected, we find two non–trivial solutions of Form Factors families. In each deformation, the FF's of the deforming operator suitably rescaled by (2.6), can be immediately identified because they satisfy the peculiar equations characterizing the trace of the energy–momentum tensor (2.8) and (2.9). This is further confirmed by employing the spectral representation of the correlator $\langle \Theta(x)\Theta(0)\rangle_c$ in the sum–rule (1.5), which provides in both deformations the value of the central charge with a very high precision (the relative error being of order 10^{-4}–10^{-5}). The identification of both the solutions with the primaries $\phi_{1,2}$ and $\phi_{1,3}$ is easily established after computing for each solution its UV anomalous dimension by means of the sum rule (1.4). The contributions to this sum rule coming from the dominant lightest multiparticle states are given in Tables 5 and 6 for the two deformations (the contributions are ordered according to increasing values of the Mandelstam variable s of the multi–particle state). The agreement of the truncated sums with the known values of the anomalous dimensions is very satisfactory given the fast convergency behaviour of the spectral series. In the computation of these sum rules, some three–particle FF contributions have been inserted as well, although we do give here their exact expression for sake of simplicity (their general parameterization follows the one adopted for instance, in [3]). It should be noticed that the oscillating behaviour of these sums is typical of non–unitary theories where one expects, in general, both positive and negative terms.

3.1 Non–Integrable Deformations of $\mathcal{M}(2,7)$

For each possible integrable deformation of the model, the addition of a further orthogonal deformation breaks its integrability leading, among other things, to corrections of the mass spectrum and of the vacuum energy. Both corrections can be independently computed by performing a numerical diagonalization of the off–critical Hamiltonian by means of the so–called Truncation Method [13]. We have carried out this analysis comparing these non–integrable data with the theoretical predictions by eqs. (1.3). Let us

briefly describe the output of these studies.

The double non–integrable deformation

$$[\mathcal{M}(2,7)]_{(1,3)} + \varepsilon\phi_{1,2},$$

for small values of $\varepsilon m_1^{2\Delta_{1,2}-2}$ has already been studied in [9], where a good agreement between numerical and theoretical values has been found. Having obtained the FF's for the $\phi_{1,2}$ deformation, we are now able to complete the analysis by testing the opposite deformation

$$[\mathcal{M}(2,7)]_{(1,2)} + \varepsilon\phi_{1,3} .$$

The numerical determination of the two universal ratios of eq. (1.3) (for small values of $\varepsilon m_1^{2\Delta_{1,3}-2}$) gives $\frac{\delta m_1}{\delta m_2} = 0.675$ and $\frac{\delta\mathcal{E}_{vac}}{\delta m_1} = -0.244\, m_1^{(0)}$ with a precision estimated to be up to a few percents. This values fully agree with the computed theoretical values $\frac{\delta m_1}{\delta m_2} = 0.68404$ and $\frac{\delta\mathcal{E}_{vac}}{\delta m_1} = -0.24365\, m_1^{(0)}$ (see, for instance Figure 4 and 5 where the data relative to the ratios $\frac{\delta m_1}{\delta m_2}$ and $\frac{\delta\mathcal{E}_{vac}}{\delta m_1}$ respectively are reported for different values of ε).

4 Integrable Deformations of $\mathcal{M}(2,9)$

In this section, we turn our attention to the $\mathcal{M}(2,9)$ minimal model which displays a richer structure in the RG space of relevant couplings. This model has in fact, besides the identity, three primary operators $\phi_{1,2}$, $\phi_{1,3}$ and $\phi_{1,4}$ which are all relevant with conformal dimensions $-1/3$, $-5/9$ and $-2/3$ respectively. These fields taken separately give rise to different integrable deformations of the conformal model, each of them characterized by a different mass spectrum and S–matrix (see tables B3, B4 and B5 in Appendix B). In particular, the first two deformations produce three–particle mass spectra (with different mass ratios) while the last one gives a four–particle spectrum.

The FF's of the primary operators in the $\phi_{1,3}$–deformation had already been obtained in ref. [8] and were known to satisfy the cluster property. Again, our derivation of these FF's as solutions of the cluster equations proves that the FF's found in [8] are the only possible cluster solutions.

The Form Factors of the cluster solutions for each of the three above mentioned deformations have been computed according to the strategy explained in Section 2. The

resulting one–particle FF's and two–particle FF's coefficients are given in tables 7–8, 9–10 and 11–12 respectively. The important result is that in each integrable deformation of this model, three families of non–trivial solutions have been found. Among the solutions, we have firstly identified the FF's of the deforming field by checking the exact fulfillment of eqs. (2.8) and (2.9), after the appropriate rescaling (2.6). Moreover, the c–sum–rule (1.5) can be easily shown to give very precise approximations of the central charge in each of the three separate deformations.

As for the other solutions, they have been successfully identified with the FF's of the primary operators by computing their anomalous dimension by means of eq. (1.4). The first contributions to these sums are given in tables 13, 14 and 15. In all cases the agreement with the expected anomalous dimensions of the primaries is established, even though the convergence of the series is observed to be noticeably faster for lower absolute values of the anomalous dimension of the deforming field. This observed trend is indeed expected from the short–distance behavior of the correlator (2.10), as predicted by the Operator Product Expansion of the fields. In fact, in the models $\mathcal{M}(2, 2\,n + 1)$ where the fields have negative anomalous dimensions, this correlator displays a zero at the origin whose power law exponent is larger for lower absolute values of the anomalous dimension of $\Theta(x)$; correspondingly, the small x region of integration in (1.4) is less relevant making the lightest multiparticle states more dominant in the series.

4.1 Non–Integrable Deformations of $\mathcal{M}(2, 9)$

The availability of the FF's of all the primary fields of the model has allowed us, in each of the three separate integrable deformations, to consider two different orthogonal non–integrable deformations. We have had then the possibility of testing the theoretical values obtained for the universal quantities (1.3) versus their numerical TCS estimates in six different multiple deformations, exploring in this way the non–integrable region around the conformal point of the model. The outcome of the analysis in all the deformations is summarized in Table 16. Since the precision of TCS data is expected to be of approximately a few percents, the comparison with the computed theoretical values is in all cases quite satisfactory.

5 Conclusions

The main purpose of this work has been to substantiate by means of concrete *ab–initio* calculations the cluster hypothesis for the Form Factors of the relevant operators in integrable quantum field theories obtained as deformation of a conformal action. We have studied, in particular, the matrix elements of the primary operators in the integrable deformations of the first models of the non–unitary series $\mathcal{M}(2, 2n + 1)$. In all cases analysed, we have confirmed the cluster hypothesis since we have found a one–to–one correspondence between the independent solutions of the cluster equations and the relevant fields.

It should be said that the absence of internal symmetries of the above models has played an important role in carrying out our computations. In fact, in this situation one can exploit the cluster equations (1.6) in their full generality. It would be interesting to see how the results of ref. [10] generalize to the case of quantum field theories with internal symmetries which induce selection rules on the matrix elements. Another important open problem is also to understand the meaning of the cluster properties in quantum field theories which cannot be regarded as deformation of conformal models. A complete understanding of all these aspects of the Form Factors would allow us to better understand the asymptotic high–energy regime of quantum theories and their operator content.

Appendix A

In this appendix we give the general parameterization adopted throughout the paper for two–particle Form Factors. The S–matrices of the specific models analysed in this paper are given in Appendix B where the generic amplitude

$$S_{ab}(\theta) = \prod_{x \in P_{ab}} (x)^{p_x} \prod_{y \in Z_{ab}} (-y)^{q_y} \tag{A.1}$$

is written adopting the notation

$$(\alpha) = \frac{\tanh \frac{1}{2}(\theta + i\pi\alpha)}{\tanh \frac{1}{2}(\theta - i\pi\alpha)} \tag{A.2}$$

and the positive rational indices x and y label the poles and the zeros displayed by the amplitude in the physical strip $\text{Im}\,\theta \in [0, \pi]$. The bound state simple poles of the S–matrices are identified by superscripts which denote the particles produced. There are however also simple poles of a different nature which have been labelled with suffices $\mathcal{B}$ and $\mathcal{D}$ that are not related to any bound state and are due to multiparticle scattering processes of the kind shown in Figure 2 and 3 respectively. The fact that these diagrams (which usually produce second and third order poles) are here responsible for simple poles is due to the occurrence of zeros in the S–matrix factors carried by the internal crossing lines [20]. Higher order poles are present as well and, among these, in the model $[\mathcal{M}(2/9)]_{(1,4)}$, some triple poles labelled by an asterisk which also have a non–standard diagrammatic interpretation. The understanding of the nature of all the poles is necessary in order to assign to the FF's the correct pole structure.

The general two–particle Form Factor of a scalar operator $\Phi(x)$

$$F_{ab}^{\Phi}(\theta_1 - \theta_2) = \langle 0 \,|\, \Phi(0) \,|\, A_a(\theta_1) \, A_b(\theta_2) \rangle \,, \tag{A.3}$$

will be parameterized by

$$F_{ab}^{\Phi}(\theta) = Q_{ab}^{\Phi}(\theta) \frac{F_{ab}^{\min}(\theta)}{D_{ab}(\theta)} \,. \tag{A.4}$$

where the "minimal" FF

$$F_{ab}^{\min}(\theta) = \left(-i \sinh(\theta/2) \right)^{\frac{1 - S_{ab}(0)}{2}} \frac{\displaystyle\prod_{x \in P} g_x^{p_x}(\theta)}{\displaystyle\prod_{y \in Z_{ab}} g_y^{q_y}(\theta)} \,, \tag{A.5}$$

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which has neither zeros nor poles in the physical strip, is written in terms of the function

$$g_x(\theta) = \prod_{k=0}^{\infty} \left[\frac{\left[1 + \left[\frac{\frac{i\pi-\theta}{2\pi}}{n+\frac{1}{2}+\frac{x}{2}}\right]^2\right]\left[1 + \left[\frac{\frac{i\pi-\theta}{2\pi}}{n+1-\frac{x}{2}}\right]^2\right]}{\left[1 + \left[\frac{\frac{i\pi-\theta}{2\pi}}{n+1+\frac{x}{2}}\right]^2\right]\left[1 + \left[\frac{\frac{i\pi-\theta}{2\pi}}{n+\frac{3}{2}-\frac{x}{2}}\right]^2\right]} \right]^{k+1} . \tag{A.6}$$

This function is normalized by $g_x(i\pi) = 1$ and behaves asymptotically as

$$g_x(\theta) \sim e^{|\theta|/2} \qquad \text{for} \qquad \theta \to \infty . \tag{A.7}$$

The factor $Q_{ab}^{\Phi}(\theta)$ in (A.4) is a polynomial in $\cosh\theta$ carrying the dependence on the specific operator $\Phi(x)$

$$Q_{ab}^{\Phi} = \sum_{k=0}^{k_{ab,\Phi}^{max}} a_{ab,\Phi}^{(k)} \cosh^k(\theta) . \tag{A.8}$$

The most subtle element in the parameterization of the FF's is represented by the structure of the poles which, in eq. (A.4) are introduced by the factor $D_{ab}(\theta)$. In order to establish which poles are to be found in a FF one must in general have a complete understanding of the nature of the poles in the corresponding S–matrix element in terms of microscopical processes. We will write in general,

$$D_{ab}(\theta) = \prod_{x \in P_{ab}} \mathcal{P}_x^{i_x} \mathcal{P}_{1-x}^{j_x} , \tag{A.9}$$

where the set of indices is defined in (A.1) and

$$\mathcal{P}_x(\theta) = \frac{\cos(\pi x) - \cosh(\theta)}{2\cos^2(\frac{\pi x}{2})} . \tag{A.10}$$

For bound–state simple poles and ordinary higher order poles of the S–matrix, the correct rule for determining the indices i_x and j_x is given by [3]

$$\begin{aligned} i_x = n , & \quad j_x = n-1 , \quad \text{if} \quad p_x = 2n-1 ; \\ i_x = n , & \quad j_x = n , \quad\;\; \text{if} \quad p_x = 2n , \end{aligned} \tag{A.11}$$

For simple poles of type $(x)_{\mathcal{B}}$ and $(x)_{\mathcal{D}}$ one can show that the correct indices are still $i_x = 1$ and $j_x = 0$, as for a bound state simple pole.

Notice however that the poles of the FF's induced by the triple poles labelled with $*$ in $[\mathcal{M}(2/9)]_{(1,4)}$ do not fall within the above analysis. Their general expressions is not further investigated here since these FF's were not needed in the present work.

As a final remark, notice that every function (α) could be equivalently written as $(1 - \alpha)$ without changing the S–matrices. However, the pole prescription given above for the FF's is *sensitive* to this change in the case of odd order poles. Therefore, all the labels α in the S–matrices reported here have been chosen to give (in units of $i\pi$) the value of the direct s–channel resonant angles in the case of bound state odd poles and also in the case of poles of type $\mathcal{B}$ and $\mathcal{D}$ [7]. Only with this choice, the above prescription gives the correct poles of the FF's.

With the parameterization (A.4), the two–particle Form Factor of a general operator Φ is therefore completely determined after fixing the coefficients $a^{(k)}_{ab,\Phi}$ in the expansion (A.8).

---

[7]For s–channel in these cases we mean the one defined by Figures 2 and 3 with particles flowing upward

Appendix B

In this Appendix we give the S–matrices of the integrable models analyzed in this work. The function (α) used in the tables is given in eq. (A.2). Anomalous simple poles have been labelled with $\mathcal{B}$ and $\mathcal{D}$, while the anomalous triple poles of the model $[\mathcal{M}(2/9)]_{(1,4)}$ are identified with $*$.

$$
\begin{aligned}
S_{11}(\theta) &= \overset{1}{\left(\tfrac{2}{3}\right)} \overset{2}{\left(\tfrac{1}{9}\right)} \left(-\tfrac{2}{9}\right) \\[2mm]
S_{12}(\theta) &= \overset{1}{\left(\tfrac{17}{18}\right)} \left(\tfrac{11}{18}\right)_{\mathcal{B}} \\[2mm]
S_{22}(\theta) &= \overset{2}{\left(\tfrac{2}{3}\right)} \left(\tfrac{8}{9}\right)_{\mathcal{B}} \left(\tfrac{5}{9}\right)_{\mathcal{D}}
\end{aligned}
$$

$$
m_2 = 2 \cos \frac{\pi}{18} \, m_1 = 1.9696... \, m_1
$$

Table B1: S–Matrix and mass ratios of the $[M(2/7)]_{(1,2)}$ model.

$$
\begin{aligned}
S_{11}(\theta) &= \overset{2}{\left(\tfrac{2}{5}\right)} \\[2mm]
S_{12}(\theta) &= \overset{1}{\left(\tfrac{4}{5}\right)} \overset{2}{\left(\tfrac{3}{5}\right)} \\[2mm]
S_{22}(\theta) &= \overset{1}{\left(\tfrac{4}{5}\right)} \left(\tfrac{2}{5}\right)^{2}
\end{aligned}
$$

$$
m_2 = 2 \cos \frac{\pi}{5} \, m_1 = 1.61803\ldots \, m_1
$$

Table B2: S–Matrix and mass ratios of the $[M(2/7)]_{(1,3)}$ model.

$$\begin{aligned}
S_{11}(\theta) &= \overset{\mathbf{1}}{\left(\tfrac{2}{3}\right)} \overset{\mathbf{2}}{\left(\tfrac{1}{12}\right)} \left(-\tfrac{1}{4}\right) \\[4pt]
S_{12}(\theta) &= \overset{\mathbf{1}}{\left(\tfrac{23}{24}\right)} \overset{\mathbf{3}}{\left(\tfrac{1}{8}\right)} \left(\tfrac{5}{8}\right)_{\mathcal{B}} \left(-\tfrac{5}{24}\right) \\[4pt]
S_{13}(\theta) &= \overset{\mathbf{2}}{\left(\tfrac{11}{12}\right)} \left(\tfrac{7}{12}\right)_{\mathcal{B}} \\[4pt]
S_{22}(\theta) &= \overset{\mathbf{2}}{\left(\tfrac{2}{3}\right)} \left(\tfrac{11}{12}\right)^{2} \left(\tfrac{7}{12}\right)_{\mathcal{D}} \left(-\tfrac{1}{4}\right) \\[4pt]
S_{23}(\theta) &= \overset{\mathbf{1}}{\left(\tfrac{23}{24}\right)} \left(\tfrac{7}{8}\right)_{\mathcal{B}} \left(\tfrac{5}{8}\right)_{\mathcal{B}} \left(\tfrac{13}{24}\right)_{\mathcal{D}} \\[4pt]
S_{33}(\theta) &= \overset{\mathbf{3}}{\left(\tfrac{2}{3}\right)} \left(\tfrac{11}{12}\right)_{\mathcal{B}} \left(\tfrac{5}{6}\right)_{\mathcal{B}} \left(\tfrac{7}{12}\right)_{\mathcal{D}} \left(\tfrac{1}{2}\right)_{\mathcal{D}} \\[6pt]
\hline \\[-6pt]
m_a &= \frac{\sin\frac{a\pi}{24}}{\sin\frac{\pi}{24}} \, m_1 \qquad a = 1,\, 2,\, 3
\end{aligned}$$

Table B3: S–Matrix and mass ratios of the $[M(2/9)]_{(1,2)}$ model.

$$\begin{aligned}
S_{11}(\theta) &= \overset{\mathbf{2}}{\left(\tfrac{2}{7}\right)} \\[4pt]
S_{12}(\theta) &= \overset{\mathbf{1}}{\left(\tfrac{6}{7}\right)} \overset{\mathbf{3}}{\left(\tfrac{3}{7}\right)} \\[4pt]
S_{13}(\theta) &= \overset{\mathbf{2}}{\left(\tfrac{5}{7}\right)} \overset{\mathbf{3}}{\left(\tfrac{4}{7}\right)} \\[4pt]
S_{22}(\theta) &= \overset{\mathbf{3}}{\left(\tfrac{4}{7}\right)} \left(\tfrac{5}{7}\right)^{2} \\[4pt]
S_{23}(\theta) &= \overset{\mathbf{1}}{\left(\tfrac{6}{7}\right)} \overset{\mathbf{2}}{\left(\tfrac{5}{7}\right)} \left(\tfrac{3}{7}\right)^{2} \\[4pt]
S_{33}(\theta) &= \overset{\mathbf{1}}{\left(\tfrac{6}{7}\right)} \left(\tfrac{3}{7}\right)^{2} \left(\tfrac{5}{7}\right)^{2} \\[6pt]
\hline \\[-6pt]
m_a &= \frac{\sin\frac{a\pi}{7}}{\sin\frac{\pi}{7}} \, m_1 \qquad a = 1,\, 2,\, 3
\end{aligned}$$

Table B4: S–Matrix and mass ratios of the $[M(2/9)]_{(1,3)}$ model.

$$S_{11}(\theta) = \overset{\mathbf{1}}{\left(\tfrac{2}{3}\right)} \overset{\mathbf{2}}{\left(\tfrac{7}{15}\right)} \overset{\mathbf{3}}{\left(\tfrac{2}{15}\right)} \left(-\tfrac{1}{15}\right) \left(-\tfrac{2}{5}\right)$$

$$S_{12}(\theta) = \overset{\mathbf{1}}{\left(\tfrac{23}{30}\right)} \overset{\mathbf{3}}{\left(\tfrac{13}{30}\right)}$$

$$S_{13}(\theta) = \overset{\mathbf{1}}{\left(\tfrac{14}{15}\right)} \overset{\mathbf{2}}{\left(\tfrac{11}{15}\right)} \overset{\mathbf{4}}{\left(\tfrac{1}{5}\right)} \left(\tfrac{3}{5}\right)^2 \left(-\tfrac{2}{15}\right) \left(-\tfrac{1}{3}\right)$$

$$S_{14}(\theta) = \overset{\mathbf{3}}{\left(\tfrac{13}{15}\right)} \left(\tfrac{8}{15}\right)_{\mathcal{B}} \left(\tfrac{2}{3}\right)^2$$

$$S_{22}(\theta) = \overset{\mathbf{2}}{\left(\tfrac{2}{3}\right)} \overset{\mathbf{4}}{\left(\tfrac{1}{5}\right)} \left(\tfrac{8}{15}\right)_{\mathcal{B}}$$

$$S_{23}(\theta) = \overset{\mathbf{1}}{\left(\tfrac{5}{6}\right)} \left(\tfrac{1}{2}\right)_{\mathcal{B}} \left(\tfrac{7}{10}\right)_{\mathcal{B}} \left(\tfrac{11}{30}\right)_{\mathcal{B}}$$

$$S_{24}(\theta) = \overset{\mathbf{2}}{\left(\tfrac{9}{10}\right)} \left(\tfrac{23}{30}\right)_{\mathcal{B}} \left(\tfrac{3}{10}\right)_{\mathcal{B}} \left(\tfrac{19}{30}\right)_{\mathcal{B}} \left(\tfrac{17}{30}\right)^2$$

$$S_{33}(\theta) = \overset{\mathbf{3}}{\left(\tfrac{2}{3}\right)^3} \left(\tfrac{2}{15}\right)^2 \left(\tfrac{7}{15}\right)^2 \left(-\tfrac{1}{15}\right) \left(-\tfrac{2}{5}\right)$$

$$S_{34}(\theta) = \overset{\mathbf{1}}{\left(\tfrac{14}{15}\right)} \left(\tfrac{4}{5}\right)_{\mathcal{B}} \left(\tfrac{7}{15}\right)_{\mathcal{D}} \left(\tfrac{11}{15}\right)^2 \left(\tfrac{3}{5}\right)^3_{*}$$

$$S_{44}(\theta) = \overset{\mathbf{4}}{\left(\tfrac{2}{3}\right)^3_{*}} \left(\tfrac{8}{15}\right)^3_{*} \left(\tfrac{2}{5}\right)_{\mathcal{D}} \left(\tfrac{11}{15}\right)_{\mathcal{B}} \left(\tfrac{13}{15}\right)_{\mathcal{B}} \left(\tfrac{1}{5}\right)^2$$

$$m_2 = 2\cos\frac{7\pi}{30}\, m_1 = 1.48629...\, m_1$$

$$m_3 = 2\cos\frac{\pi}{15}\, m_1 = 1.95630...\, m_1$$

$$m_4 = 2\cos\frac{\pi}{10}\, m_2 = 2.82709...\, m_1$$

Table B5: S–Matrix and mass ratios of the $[M(2/9)]_{(1,4)}$ model.

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Table Captions

Table 1 One–particle form factors of cluster solutions in $[\mathcal{M}(2/7)]_{(1,2)}$.

Table 2 Two–particle form factors coefficients of cluster solutions in $[\mathcal{M}(2/7)]_{(1,2)}$.

Table 3 One–particle form factors of cluster solutions in $[\mathcal{M}(2/7)]_{(1,3)}$.

Table 4 Two–particle form factors coefficients of cluster solutions in $[\mathcal{M}(2/7)]_{(1,3)}$.

Table 5 Sum rules of the conformal dimensions of primary operators in $[\mathcal{M}(2/7)]_{(1,2)}$.

Table 6 Sum rules of the conformal dimensions of primary operators in $[\mathcal{M}(2/7)]_{(1,2)}$.

Table 7 One–particle form factors of cluster solutions in $[\mathcal{M}(2/9)]_{(1,2)}$.

Table 8 Two–particle form factors coefficients of cluster solutions in $[\mathcal{M}(2/9)]_{(1,2)}$.

Table 9 One–particle form factors of cluster solutions in $[\mathcal{M}(2/9)]_{(1,3)}$.

Table 10 Two–particle form factors coefficients of cluster solutions in $[\mathcal{M}(2/9)]_{(1,3)}$.

Table 11 One–particle form factors of cluster solutions in $[\mathcal{M}(2/9)]_{(1,4)}$.

Table 12 Two–particle form factors coefficients of cluster solutions in $[\mathcal{M}(2/9)]_{(1,4)}$.

Table 13 Sum rules of the conformal dimensions of primary operators in $[\mathcal{M}(2/9)]_{(1,2)}$.

Table 14 Sum rules of the conformal dimensions of primary operators in $[\mathcal{M}(2/9)]_{(1,3)}$.

Table 15 Sum rules of the conformal dimensions of primary operators in $[\mathcal{M}(2/9)]_{(1,4)}$.

Table 16 Comparison between numerical and theoretical estimates of data obtained in different non–integrable deformations of $\mathcal{M}_{(2,9)}$. .

$\mathcal{O}$	$\phi_{1,2}$	$\phi_{1,3}$
$F_1^{\mathcal{O}}$	$0.8129447456\,i$	$1.245503611\,i$
$F_2^{\mathcal{O}}$	-0.1200387686	-0.4656766285

Table 1

$\mathcal{O}$	$\phi_{1,2}$	$\phi_{1,3}$
$a_{11,\mathcal{O}}^{(0)}$	-0.6905355776	-0.4178217785
$a_{11,\mathcal{O}}^{(1)}$	1.570496171	3.686419944
$a_{12,\mathcal{O}}^{(0)}$	$31.91217166\,i$	$160.8200658\,i$
$a_{12,\mathcal{O}}^{(1)}$	$25.76337182\,i$	$153.1262244\,i$
$a_{22,\mathcal{O}}^{(0)}$	-12.74804909	-71.66459155
$a_{22,\mathcal{O}}^{(1)}$	-3.97663589	-59.84689851

Table 2

$\mathcal{O}$	$\phi_{1,2}$	$\phi_{1,3}$
$F_1^{\mathcal{O}}$	$0.8703387193\,i$	$1.408237641\,i$
$F_2^{\mathcal{O}}$	-0.3322661173	-0.8698840033

Table 3

$\mathcal{O}$	$\phi_{1,2}$	$\phi_{1,3}$
$a_{11,\mathcal{O}}^{(0)}$	-1.453085043	-3.804226098
$a_{12,\mathcal{O}}^{(0)}$	$10.38924846\,i$	$30.40986050\,i$
$a_{12,\mathcal{O}}^{(1)}$	$6.420908640\,i$	$27.19940617\,i$
$a_{22,\mathcal{O}}^{(0)}$	-13.76381909	-42.18951412
$a_{22,\mathcal{O}}^{(1)}$	-4.702281947	-32.22992104

Table 4

states	s	Δ_{12}–terms	Δ_{13}–terms
A_1	$1.000\,m_1$	-0.2922910	-0.4478157
A_2	$1.969\,m_1$	0.0016428	0.0063729
$A_1\ A_1$	$\geq 2.000\,m_1$	0.0051123	0.0137590
$A_1\ A_2$	$\geq 2.969\,m_1$	-0.0000763	-0.0004400
$A_1\ A_1\ A_1$	$\geq 3.000\,m_1$	-0.0001040	-0.0004777
$A_2\ A_2$	$\geq 3.939\,m_1$	0.0000003	0.0000040
sum		-0.2857159	-0.4285976
value expected		-0.2857143	-0.4285714

Table 5

states	s	Δ_{12}–terms	Δ_{13}–terms
A_1	$1.000\,m_1$	-0.3221795	-0.5212974
A_2	$1.618\,m_1$	0.0290206	0.0759768
$A_1\ A_1$	$2.000\,m_1$	0.0098699	0.0258398
$A_1\ A_2$	$2.618\,m_1$	-0.0023149	-0.0089996
$A_1\ A_1\ A_1$	$3.000\,m_1$	-0.0003334	-0.0013803
$A_2\ A_2$	$3.236\,m_1$	0.0001155	0.0006612
sum		-0.2858218	-0.4291998
value expected		-0.2857143	-0.4285714

Table 6

$\mathcal{O}$	$\phi_{1,2}$	$\phi_{1,3}$	$\phi_{1,4}$
$F_1^{\mathcal{O}}$	$0.7548301717\,i$	$1.288575652\,i$	$1.564862744\,i$
$F_2^{\mathcal{O}}$	-0.1056909725	-0.4593398099	-0.7331609072
$F_3^{\mathcal{O}}$	$-0.01375684037\,i$	$-0.1175389994\,i$	$-0.2854817817\,i$

Table 7

$\mathcal{O}$	$\phi_{1,2}$	$\phi_{1,3}$	$\phi_{1,4}$
$a_{11,\mathcal{O}}^{(0)}$	-0.3810248990	0.1280888115	0.6449629545
$a_{11,\mathcal{O}}^{(1)}$	1.289925788	3.759118917	5.543942595
$a_{12,\mathcal{O}}^{(0)}$	$14.10905183\,i$	$75.18632019\,i$	$110.3472056\,i$
$a_{12,\mathcal{O}}^{(1)}$	$-12.74323779\,i$	$-79.90895489\,i$	$-180.9845092\,i$
$a_{12,\mathcal{O}}^{(2)}$	$-19.36998044\,i$	$-143.7096872\,i$	$-278.5592522\,i$
$a_{13,\mathcal{O}}^{(0)}$	-1.826322080	-18.97540047	-51.56786333
$a_{13,\mathcal{O}}^{(1)}$	-1.116015559	-16.27774386	-48.01279071
$a_{22,\mathcal{O}}^{(0)}$	-1.466545085	-3.003367424	14.9160654
$a_{22,\mathcal{O}}^{(1)}$	7.821352950	60.49540624	160.4007705
$a_{22,\mathcal{O}}^{(2)}$	2.717967823	51.33773403	130.7877664
$a_{23,\mathcal{O}}^{(0)}$	$153.8279467\,i$	$1842.946063\,i$	$5426.66381\,i$
$a_{23,\mathcal{O}}^{(1)}$	$175.5584268\,i$	$2962.508857\,i$	$9796.436391\,i$
$a_{23,\mathcal{O}}^{(2)}$	$30.43124786\,i$	$1130.002086\,i$	$4380.673323\,i$
$a_{33,\mathcal{O}}^{(0)}$	-32.42110324	-450.0936155	-1394.808207
$a_{33,\mathcal{O}}^{(1)}$	-20.23293766	-589.1376530	-2309.626757
$a_{33,\mathcal{O}}^{(2)}$	-2.174915595	-158.7701993	-936.6165096

Table 8

$\mathcal{O}$	$\phi_{1,2}$	$\phi_{1,3}$	$\phi_{1,4}$
$F_1^{\mathcal{O}}$	$0.8020765716\,i$	$1.445292066\,i$	$1.802249672\,i$
$F_2^{\mathcal{O}}$	-0.3139111339	-1.019263084	-1.584911324
$F_3^{\mathcal{O}}$	$-0.1373692453\,i$	$-0.5561967434\,i$	$-1.002231818\,i$

Table 9

$\mathcal{O}$	$\phi_{1,2}$	$\phi_{1,3}$	$\phi_{1,4}$
$a_{11,\mathcal{O}}^{(0)}$	-0.9631492344	-3.127326026	-4.862860736
$a_{12,\mathcal{O}}^{(0)}$	$10.64696613\,i$	$40.73951464\,i$	$72.35568181\,i$
$a_{12,\mathcal{O}}^{(1)}$	$5.908620424\,i$	$34.57048356\,i$	$67.03219861\,i$
$a_{13,\mathcal{O}}^{(0)}$	-2.592348236	-11.32977918	-21.24912975
$a_{13,\mathcal{O}}^{(1)}$	-1.153703500	-8.417302355	-18.91350458
$a_{22,\mathcal{O}}^{(0)}$	-5.978990567	-26.44069921	-49.87674876
$a_{22,\mathcal{O}}^{(1)}$	-1.544771430	-16.28633559	-39.37864116

Table 10

$\mathcal{O}$	$\phi_{1,2}$	$\phi_{1,3}$	$\phi_{1,4}$
$F_1^{\mathcal{O}}$	$-0.9043544898\,i$	$-1.72785339\,i$	$-2.211259663\,i$
$F_2^{\mathcal{O}}$	-0.5483648961	-1.476188315	-2.169493373
$F_3^{\mathcal{O}}$	$0.2673316508\,i$	$0.8709319528\,i$	$1.45902371\,i$
$F_4^{\mathcal{O}}$	-0.08488118964	-0.3489749771	-0.6451795597

Table 11

$\mathcal{O}$	$\phi_{1,2}$	$\phi_{1,3}$	$\phi_{1,4}$
$a_{11,\mathcal{O}}^{(0)}$	1.623982681	4.256426530	6.219867507
$a_{11,\mathcal{O}}^{(1)}$	-0.9778411563	-1.325966569	-1.477684504
$a_{11,\mathcal{O}}^{(2)}$	-2.029027259	-7.406691607	-12.13081476
$a_{12,\mathcal{O}}^{(0)}$	$-9.935037127\,i$	$-30.48935000\,i$	$-50.02403946\,i$
$a_{12,\mathcal{O}}^{(1)}$	$-4.790105254\,i$	$-24.63686052\,i$	$-46.33773309\,i$
$a_{13,\mathcal{O}}^{(0)}$	-45.21074197	-145.9600730	-220.9609710
$a_{13,\mathcal{O}}^{(1)}$	-441.3756086	-1583.189947	-2731.357697
$a_{13,\mathcal{O}}^{(2)}$	-533.3237140	-2301.408527	-4364.089257
$a_{13,\mathcal{O}}^{(3)}$	-139.4173540	-867.7984505	-1860.500753
$a_{14,\mathcal{O}}^{(0)}$	$44.34032961\,i$	$189.6077348\,i$	$357.8357762\,i$
$a_{14,\mathcal{O}}^{(1)}$	$54.79194008\,i$	$275.9507675\,i$	$562.9638801\,i$
$a_{14,\mathcal{O}}^{(2)}$	$11.43395882\,i$	$89.81474554\,i$	$212.5038539\,i$
$a_{22,\mathcal{O}}^{(0)}$	-9.190266093	-30.91094092	-52.20534546
$a_{22,\mathcal{O}}^{(1)}$	-2.709639668	-19.63612453	-42.41201502
$a_{23,\mathcal{O}}^{(0)}$	$-81.75802420\,i$	$-304.2244838\,i$	$-530.8872409\,i$
$a_{23,\mathcal{O}}^{(1)}$	$-92.61128143\,i$	$-446.8601169\,i$	$-884.9723034\,i$
$a_{23,\mathcal{O}}^{(2)}$	$-16.66533965\,i$	$-146.1571720\,i$	$-359.8444599\,i$

Table 12

states	s	Δ_{12}–terms	Δ_{13}–terms	Δ_{14}–terms
A_1	$1.000\,m_1$	-0.3409847	-0.5820972	-0.7069063
A_2	$1.982\,m_1$	0.0017003	0.0073894	0.0117945
$A_1\ A_1$	$\geq 2.000\,m_1$	0.0061957	0.0207909	0.0316698
A_3	$2.931\,m_1$	-0.0000132	-0.0001126	-0.0002734
$A_1\ A_2$	$\geq 2.982\,m_1$	-0.0000951	-0.0007084	-0.0014392
$A_1\ A_1\ A_1$	$\geq 3.000\,m_1$	-0.0001421	-0.0009038	-0.0017386
$A_1\ A_3$	$\geq 3.931\,m_1$	0.0000009	0.0000117	0.0000339
$A_2\ A_2$	$\geq 3.965\,m_1$	0.0000004	0.0000061	0.0000157
$A_2\ A_3$	$\geq 4.914\,m_1$	-0.0000000	-0.0000002	-0.0000008
sum		-0.3333379	-0.5556241	-0.6668445
value expected		-0.3333333	-0.5555556	-0.6666667

Table 13

states	s	Δ_{12}–terms	Δ_{13}–terms	Δ_{14}–terms
A_1	$1.000\,m_1$	-0.370679	-0.667941	-0.832909
A_2	$1.802\,m_1$	0.031509	0.102310	0.159088
$A_1\ A_1$	$\geq 2.000\,m_1$	0.013898	0.045127	0.070170
A_3	$2.247\,m_1$	-0.004839	-0.019592	-0.035304
$A_1\ A_2$	$\geq 2.802\,m_1$	-0.003604	-0.018722	-0.035573
$A_1\ A_1\ A_1$	$\geq 3.000\,m_1$	-0.000628	-0.003514	-0.006763
$A_1\ A_3$	$\geq 3.247\,m_1$	0.000663	0.004114	0.008844
$A_2\ A_2$	$\geq 3.604\,m_1$	0.000211	0.001684	0.003864
sum		-0.333469	-0.556534	-0.668583
value expected		-0.333333	-0.555556	-0.666667

Table 14

states	s	Δ_{12}–terms	Δ_{13}–terms	Δ_{14}–terms
A_1	$1.000\,m_1$	-0.451081	-0.861833	-1.102950
A_2	$1.486\,m_1$	0.121478	0.327017	0.480603
A_3	$1.956\,m_1$	-0.022989	-0.074895	-0.125468
$A_1\ A_1$	$\geq 2.000\,m_1$	0.035896	0.121577	0.197637
$A_1\ A_2$	$\geq 2.486\,m_1$	-0.023279	-0.101138	-0.183618
A_4	$2.827\,m_1$	0.001546	0.006354	0.011748
$A_1\ A_3$	$\geq 2.956\,m_1$	0.004304	0.022374	0.045474
$A_2\ A_2$	$\geq 2.973\,m_1$	-0.001535	-0.009929	-0.022429
$A_2\ A_3$	$\geq 3.443\,m_1$	-0.000330	-0.002101	-0.004686
$A_1\ A_4$	$\geq 3.827\,m_1$	0.003595	0.020054	0.040870
sum		-0.332396	-0.552519	-0.662819
value expected		-0.333333	-0.555556	-0.666667

Table 15

	$\dfrac{\delta m_1}{\delta m_2}$		$\dfrac{\delta\mathcal{E}_{vac}}{m_1^{(0)}\delta m_1}$	
deformation	numerical ($\pm3\%$)	theoretical	numerical ($\pm3\%$)	theoretical
$[\mathcal{M}(2,9)]_{(1,2)} + \varepsilon\phi_{1,3}$	0.590	0.592049	-0.275	-0.275404
$[\mathcal{M}(2,9)]_{(1,2)} + \varepsilon\phi_{1,4}$	0.661	0.660963	-0.204	-0.204124
$[\mathcal{M}(2,9)]_{(1,3)} + \varepsilon\phi_{1,2}$	0.390	0.391396	-1.04	-1.03826
$[\mathcal{M}(2,9)]_{(1,3)} + \varepsilon\phi_{1,4}$	0.811	0.83681	-0.205	-0.205640
$[\mathcal{M}(2,9)]_{(1,4)} + \varepsilon\phi_{1,2}$	-0.133	-0.131367	1.73	1.74582
$[\mathcal{M}(2,9)]_{(1,4)} + \varepsilon\phi_{1,3}$	0.238	0.240486	-0.550	-0.548156

Table 16

Figure Captions

Figure 1 Bound–state simple pole diagram.

Figure 2 "Butterfly" diagram.

Figure 3 "Dragonfly" diagram.

Figure 4 Numerical TCS estimates of δm_1 versus δm_2 for different values of the "non–integrable" coupling in the model $[\mathcal{M}(2/7)]_{(1,2)} + \varepsilon\phi_{1,3}$. The dashed line represents the theoretical prediction.
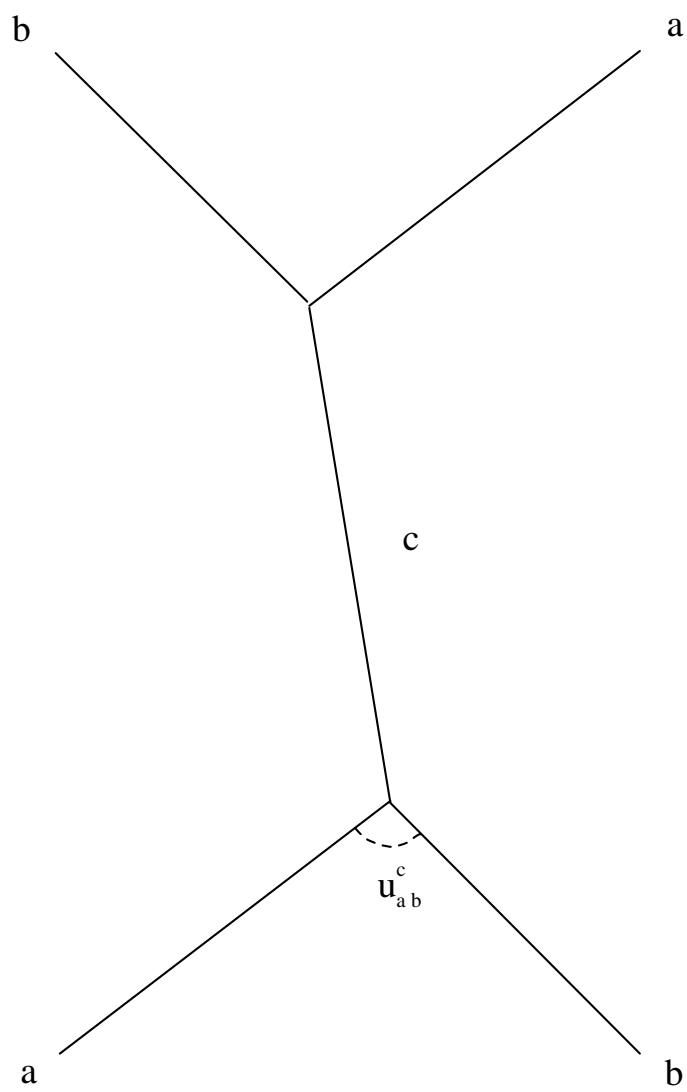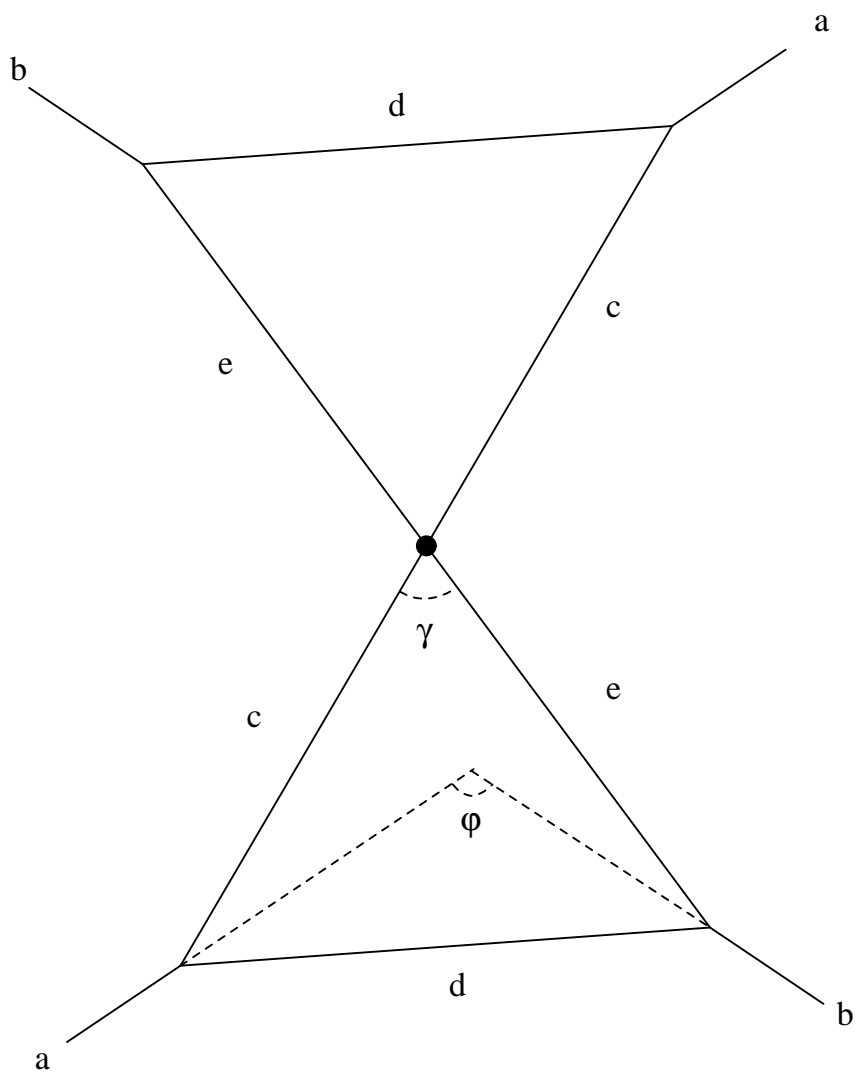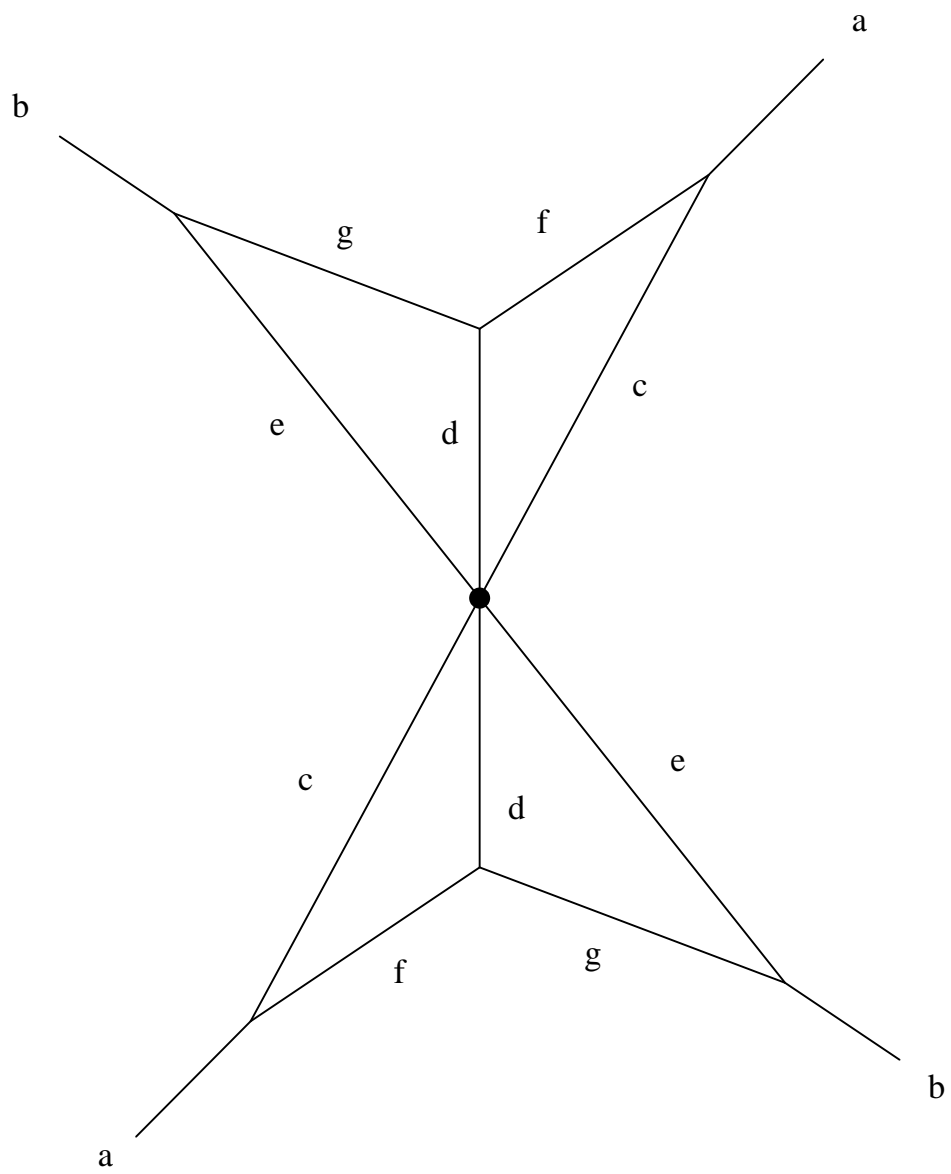
Figure 5 Numerical TCS estimates of $\delta\mathcal{E}_{vac}$ versus $m_1^{(0)}\delta m_1$ for different values of the "non–integrable" coupling in the model $[\mathcal{M}(2/7)]_{(1,2)} + \varepsilon\phi_{1,3}$. The dashed line represents the theoretical prediction.
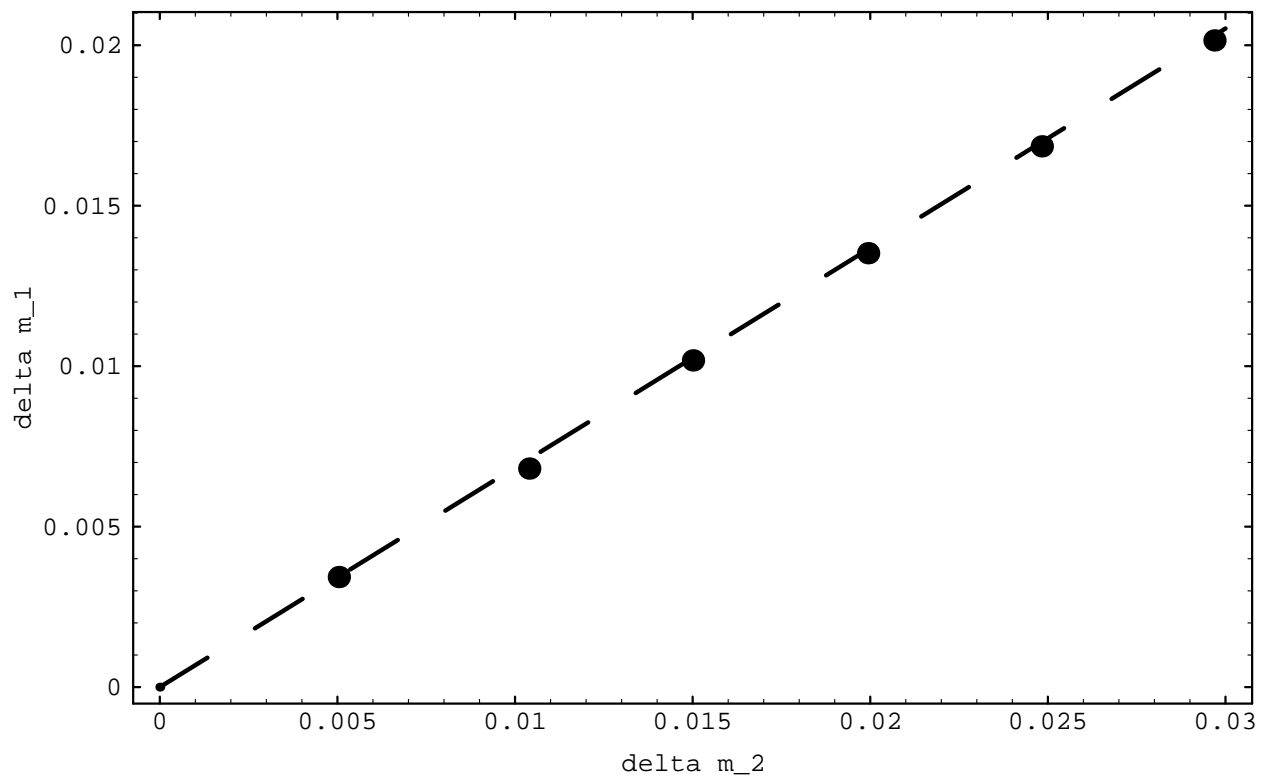
Figure 1

Figure 2

Figure 3

Figure 4

Figure 5